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Uniqueness and Successive Approximations for Functional Differential Equations*

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1. INTRODUCTION

Differential equations which express $y'(t)$ as a function of past and present values of $y(t)$ have been called delay functional differential equations. Many of the ideas from the theory of ordinary differential equations have been generalized for this type of equation including the basic ideas of existence and uniqueness of solutions of initial value problems.

Let α and t_0 be numbers where $-\infty \leqslant \alpha < \infty$ and t_0 ($\geqslant \alpha$) is a finite number. In case $\alpha = -\infty$ read $[\alpha, t]$ as $(\alpha, t]$. Suppose that $\phi(t)$ is a prescribed continuous n-vector function on $[\alpha, t_0]$ and we wish to find a continuous function $y(t)$ on some interval $[\alpha, t_0 + a]$, $a > 0$, such that

$$\begin{aligned} y(t) &= \phi(t), & \alpha \leqslant t \leqslant t_0, \\ y'(t) &= \mathcal{F}(t, y(\cdot)), & t_0 < t \leqslant t_0 + a, \end{aligned} \tag{1}$$

where $\mathcal{F}(t, \psi(\cdot))$ is a function (functional) defined for t in $[t_0, t_0 + a]$ and ψ in $C(t)$ and taking values in $\mathfrak{R}^n$ and where $C(t)$ is all continuous functions from $[\alpha, t]$ into some set D in $\mathfrak{R}^n$. It is known [1]–[3] that for appropriate sets D if $\mathcal{F}$ is continuous and satisfies a Lipschitz condition in ψ, then the initial value problem has a unique solution and this solution may be constructed by successive approximations.

In this paper we give theorems analogous to the Nagumo and Wintner theorems for ordinary differential equations [4] which prove the solution of the initial value problem (1) for other conditions on $\mathcal{F}$ is unique and can be constructed by successive approximations.

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2. A Differential Inequality

First we give a lemma on differential inequalities which is analogous to the Kamke differential inequality theorems for ordinary differential equations [4], [5]. Let D be a region in $\mathcal{R}^n$ and let $\mathcal{F}$ be a function taking values in $\mathcal{R}^n$ for $t_0 \leqslant t \leqslant t_0 + a$ and ψ in $C_D(t)$, where $C_D(t)$ is all continuous functions from $[\alpha, t]$ into D, with the properties that $\lim_{n\to\infty} \mathcal{F}(t_n, \psi_n(\cdot)) = \mathcal{F}(t, \psi(\cdot))$ whenever $t, t_1, \ldots$ are in $[t_0, t_0 + a]$, $\psi, \psi_1 \cdots$ belong to $C_D(t_0 + a)$ and $t_n \to t, \psi_n \to \psi$ (in the sup norm topology). (If $\alpha = -\infty$, $C_D(t)$ is all continuous functions from $(-\infty, t]$ into some compact subset of D.) Further, let (t, ψ_1), (t, ψ_2) in the domain of $\mathcal{F}$ and $\psi_1(s) \leqslant \psi_2(s)$ for $\alpha \leqslant s \leqslant t$ imply $\mathcal{F}(t, \psi_1(\cdot)) \leqslant \mathcal{F}(t, \psi_2(\cdot))$. Here a vector inequality $<$ means that every component of the left vector is less than the corresponding component of the right vector, etc. Also note that we use the same notation for a function ψ given on an interval $[\alpha, t]$ and its restriction to a subinterval $[\alpha, t'] \subset [\alpha, t]$.

Lemma 1. *Let z be a continuous function from $[\alpha, t_0 + a]$ into D and be such that*

$$D^+z(t) < \mathcal{F}(t, z(\cdot)), \qquad t_0 \leqslant t < t_0 + a; \tag{2}$$

and let $y(t)$ be a continuous function with the properties

(a) $y(t) \geqslant z(t), \qquad \alpha \leqslant t \leqslant t_0,$

(b) $y'(t) = \mathcal{F}(t, y(\cdot)), \qquad t_0 \leqslant t \leqslant t_0 + a.$

Then $y(t) \geqslant z(t)$ for $t_0 < t \leqslant t_0 + a$. (Here

$$D^+z(t) = \limsup \frac{z(t+h) - z(t)}{h}$$

as $h \to 0^+$.)

Proof. Suppose $y(t) \geqslant z(t)$ on some largest interval $[t_0, \delta]$ in $[t_0, t_0 + a]$. (Here δ could be t_0.) If for some $k \in \{1, 2,\ldots, n\}$ $z_k(\delta) = y_k(\delta)$ then

$$D^+z_k(\delta) < \mathcal{F}_k(\delta, z(\cdot)) \leqslant \mathcal{F}_k(\delta, y(\cdot)) = y_k'(\delta),$$

which implies $z(t) \leqslant y(t)$ for values of t in a right neighborhood of δ, thus $\delta = t_0 + a$.

The monotonic behavior of $\mathcal{F}$ is crucial since $y(t) = 2^{3/4}t^{1/2}$ is a solution of $y'(t) = 1/y(t/2)$ on $1 \leqslant t < 4^{1/7}$ and $z(t) = t^4$ satisfies (2) on $1 \leqslant t < 4^{1/7}$; however, $z(t) > y(t)$ for $2^{3/14} < t < 4^{1/7}$. Thus, the Kamke differential inequality type theorem requires additional conditions. The continuity condition on $\mathcal{F}$ is sufficient to guarantee the existence of at least one solution to (1), see [1], [2]. The differential inequality theorem above then will infer the existence of a right maximal solution to (1) as Coppel [5] does for ordinary differential equations. Then the strict inequality in (2) may be replaced with $\leqslant$ provided $y(t)$ is the maximal solution of (1), see Coppel [5].

3. A Uniqueness Theorem

Let ϕ be a given continuous n vector valued function on $[\alpha, t_0]$ and let $C_b(t)$ be the collection of continuous functions $f(s)$ on $[\alpha, t]$ which agree with ϕ on $[\alpha, t_0]$ and which lie in $\bar{S}(\phi(t_0), b)$ for $t_0 < s \leqslant t \leqslant t_0 + a$,

$$(\bar{S}(y_0, b) = \{y : | y - y_0 | \leqslant b\}).$$

Let $\mathcal{F}$ be a function taking values in $\mathcal{R}^n$ for t in $[t_0, t_0 + a]$ and ψ in $C_b(t)$ with the property that $\lim_{n\to\infty} \mathcal{F}(t_n, \psi_n(\cdot)) = \mathcal{F}(t, \psi(\cdot))$ whenever t, t_1,..., are in $[t_0, t_0 + a]$, ψ, ψ_1,..., belong to $C_b(t_0 + a)$ and $t_n \to t$, $\psi_n \to \psi$ (in the sup norm topology).

In addition to the continuity requirement above, we want to impose a restriction on $\mathcal{F}$ which is analogous to the Nagumo condition. For this purpose we define a set of linear functionals. Let W be the set of all linear functionals w, of the type

$$w(t, x(\cdot)) = \int_{\alpha}^{t_0+a} x(s)\, d_s\eta(t, s),$$

defined for $t_0 < t \leqslant t_0 + a$, x a continuous function on $[\alpha, t_0 + a]$ where $\eta(t, s)$ is a described below and where the functional differential equation

$$\begin{aligned} x(t) &= 0, & \alpha \leqslant t \leqslant t_0, \\ x(t)' &= w(t, x(\cdot)), & t_0 < t \leqslant t_0 + a, \end{aligned} \tag{3}$$

has a solution $f_w(t)$ on some interval $[\alpha, t_0 + \delta_w]$, where $\delta_w > 0$, such that f_w does not vanish on $(t_0, t_0 + \delta_w]$ and $\lim_{t\to t_0+} f_w(t)/t - t_0 \neq 0$. Here $\eta(t, s)$ is

(i) defined, real-valued on $(t_0, t_0 + a] \times [\alpha, t_0 + a]$,

(ii) constant for $t \leqslant s$,

(iii) nondecreasing in s,

(iv) continuous in t uniformly with respect to s. (4)

Some examples of such linear functionals w in which $f_w(t) = t^p$, $0 < p \leqslant 1$ are

(a) $w(t, x(\cdot)) = 2/t^2 \int_{t-\epsilon}^{t} x(s)\, ds$ where $\alpha = -\epsilon$, $t_0 = 0$, $\delta_w = \epsilon$.

(b) $w(t, x(\cdot)) = 1/t\, [Ax(t) + Bx(t/2)]$ where $\alpha = t_0 = 0$, A, B, such that $p = A + B/2^p$, and

(c) $w(t, x(\cdot)) = 1/t^2\, x(t^2)$ where $\alpha = t_0 = 0$.

THEOREM 1. *Let $\mathcal{F}$ be as specified above and satisfy*

$$| \mathcal{F}(t, \psi_1(\cdot)) - \mathcal{F}(t, \psi_2(\cdot)) | \leqslant w(t, | (\psi_1 - \psi_2) | (\cdot)), \tag{5}$$

for (t, ψ_1), (t, ψ_2) in the domain of $\mathcal{F}$, for $t > t_0$ and for some w in W. Then the initial value problem (1) *has at most one solution on any interval $[\alpha, t_0 + \epsilon]$ for small $\epsilon > 0$.*

PROOF. Assume there are two different solutions $y_1(t)$ and $y_2(t)$ on some interval $[\alpha, t_0 + \epsilon]$. By shrinking ϵ we may suppose that for

$$y(t) = y_1(t) - y_2(t), \qquad | y(t_0 + \epsilon) | \neq 0 \qquad \text{and} \qquad \epsilon \leqslant \delta_w .$$

Consider the solution $x(t) = 0$, $\alpha \leqslant t \leqslant t_0$

$$x(t) = \frac{1}{2} | y(t_0 + \epsilon) | \frac{f_w(t)}{| f_w(t_0 + \epsilon) |}, \qquad t_0 < t \leqslant t_0 + \epsilon$$

of the initial value problem (3). By (5) we have

$$D^+ | y(t) | \leqslant w(t, | y(\cdot) |), \qquad t_0 < t \leqslant t_0 + \epsilon.$$

For any $0 < \sigma < \epsilon$ the initial value problem

$$z(t) = \begin{cases} 0, & \alpha \leqslant t \leqslant t_0 , \\ x(t), & t_0 \leqslant t \leqslant t_0 + \sigma, \end{cases}$$

$$z'(t) = w(t, z(t)), \qquad t_0 + \sigma < t \leqslant t_0 + \epsilon,$$

has a unique solution, namely $x(t)$, since $w(t, x(\cdot))$ satisfies a Lipschitz condition in x for t in $[t_0 + \sigma, t_0 + \epsilon]$. The remarks following Lemma 1 then imply that we cannot have $x(t) > | y(t) |$ on a right neighborhood of t_0. Hence, there is a decreasing sequence of points $t_k \to t_0$ such that

$$| y(t_k) | \geqslant x(t_k), \qquad k = 1, 2, \ldots .$$

Thus $x(t_k)/(t_k - t_0) \to 0$ but this is impossible since $\lim f_w(t)/t - t_0 \neq 0$ as $t \to t_0^+$. Hence $y(t_0 + \epsilon) = 0$.

REMARK 1. The proof of the theorem makes use of the linearity of the functional w and the form of the solutions of (3). It is not readily apparent how one would generalize Kamke's uniqueness theorem [4] since if $w(t, x(\cdot))$ is a nonlinear functional, a solution of (1) passing through $\frac{1}{2} | y(t_0 + \epsilon) |$ may not exist.

REMARK 2. It is possible to relax the requirements on the integrator function $\eta(t, s)$ in the case of nth order scalar functional differential equations as shown by Theorem 3 below.

4. Successive Approximations

Next we wish to show that the sequence of successive approximations given by

$$
\begin{aligned}
&y_n(t) = \phi(t), \qquad \alpha \leqslant t \leqslant t_0, \qquad n = 1, 2, \ldots, \\
&\left.\begin{aligned} &y_1(t) = \phi(t_0), \\ &y_n(t) = \phi(t_0) + \int_{t_0}^{t} \mathscr{F}(s, y_{n-1}(\cdot))\, ds, \end{aligned}\right\} \quad \begin{aligned} &t_0 < t \leqslant t_0 + \beta, \\ &n = 1, 2, \ldots, \end{aligned}
\end{aligned} \tag{6}
$$

converges for t in $[\alpha, t_0 + \beta]$ to a solution of (1) where $\beta = \min\{a, (b/M), \delta_w\}$ and where M is such that $| \mathscr{F}(t, \psi(\cdot)) | \leqslant M$ when $t_0 \leqslant t \leqslant t_0 + \min\{a, \delta_w\}$ and ψ is in $C_b(t_0 + \min\{a, \delta_w\})$. It is easy to see that (6) defines a sequence.

Theorem 2. *Let $\mathscr{F}$ be as specified above and satisfy*

$$| \mathscr{F}(t, \psi_1(\cdot)) - \mathscr{F}(t, \psi_2(\cdot)) | \leqslant w(t, | \psi_1 - \psi_2 | (\cdot)),$$

for (t, ψ_1), (t, ψ_2) in the domain of $\mathscr{F}$ and $t > t_0$, for some w in W. Then the sequence (5) *converges to the solution of* (1) *on* $[t_0, t_0 + \beta]$.

Proof. The sequence (6) is uniformly bounded and uniformly equicontinuous on $[t_0, t_0 + \beta]$ so by Ascoli's theorem there is a subsequence $\{y_{n_k}(t)\}_{k=1}^{\infty}$ converging to some function $y(t)$ uniformly on $[t_0, t_0 + \beta]$. Hence the sequence $\{y_{n_k+1}(t)\}_{k=1}^{\infty}$ converges uniformly to some function $y^*(t)$ on $[t_0, t_0 + \beta]$. If we can show $\lambda(t) \equiv 0$, where

$$\lambda(t) = \limsup_{n\to\infty} | \omega_n(t) |, \qquad \omega_n(t) = y_n(t) - y_{n-1}(t),$$

then $y = y^* =$ the unique solution of (1) and thus the whole sequence $\{y_n(t)\}_{n=1}^{\infty}$ converges to $y(t)$ uniformly on $[t_0, t_0 + \beta]$ (see Hartman [4], pp. 4 and 41). The remainder of the proof amounts to first showing $D^+\lambda(t) \leqslant w(t, \lambda(t))$. Then if we suppose $\lambda(t) \not\equiv 0$ we may repeat the strategy in Theorem 1 to get a contradiction. The details in the case of ordinary differential equations are given in [6] and [7].

5. A Uniqueness Theorem for Scalar Problems

Let ϕ be a real-valued function with $n - 1$ continuous derivatives defined on $[\alpha, t_0]$; and for t in $[t_0, t_0 + a]$ let $C_b(t)$ be the set of all real-valued functions f with $n - 1$ continuous derivatives defined on $[\alpha, t]$ which agree with ϕ on $[\alpha, t_0]$ and satisfy

$$| f^{(k)}(t) - \phi^{(k)}(t_0) | \leqslant b, \qquad k = 0, 1, \ldots, n - 1,$$

t in $(t_0, t_0 + a]$, ($f^{(k)}$ is the kth derivative of f). Let $U(t, x(\cdot))$ be real-valued for t in $[t_0, t_0 + a]$ and x in $C_b(t)$ and be such that

$$\lim_{n \to \infty} U(t_n, \psi_n(\cdot)) = U(t, \psi(\cdot))$$

whenever $t, t_1, t_2, \ldots$ is in $[t_0, t_0 + a]$, $\psi, \psi_1, \ldots$ belong to $C_b(t_0 + a)$ and $t_n \to t$, $\psi_n \to \psi$ (in the sup norm topology).

An nth order initial value problem for delay functional differential equations is to find a function y in $C_b(t_0 + a)$ such that

$$y^{(n)}(t) = U(t, y(\cdot)), \qquad t_0 < t \leqslant t_0 + a. \tag{7}$$

Of course this type of problem is a special case of the initial value problem (1); however in this form it is possible to give new hypothesis on U so that (7) has at most one solution on any interval $[t_0, t_0 + \epsilon]$ for small $\epsilon > 0$. The following theorem generalizes a result due to Wintner [8] which is given in expanded form in Hartman [4], page 34.

THEOREM 3. *Let $\eta_k(t, s)$, $k = 0, 1, 2, \ldots, n - 1$ satisfy conditions* (i)–(iii) *of* (4) *and let $U(t, x(\cdot))$ satisfy*

$$| U(t, x(\cdot)) - U(t, y(\cdot)) | \leqslant \sum_{k=0}^{n-1} \lambda_k(t) \int_{t_0}^{t_0+a} | x^{(k)}(s) - y^{(k)}(s) | \, d_s\eta_k(t, s)$$

for t in $(t_0, t_0 + a]$, where the $\eta_k(t)$ are nonnegative functions such that

$$\sum_{k=0}^{n-1} \lambda_k(t) \int_{t_0}^{t_0+a} \frac{(s - t_0)^{n-k}}{(n - k)!} \, d_s\eta_k(t, s) \leqslant 1, \qquad t_0 < t \leqslant t_0 + a.$$

Then there is at most one solution to (7) *on any interval* $[\alpha_0, t_0 + \epsilon]$ *for small* $\epsilon > 0$.

The proof is an easy extension of the proof given in Hartman [4], page 558.

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